Dark soliton in a disorder potential

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We consider N_0 bosonic atoms in a 1D box potential of length L at zero temperature. Single particle state ϕ_0 is a solution of the Gross-Pitaevskii equation (GPE)

$$-\frac{\hbar^2}{2m}\partial_x^2\phi_0 + g_0|\phi_0|^2\phi_0 = \mu_0\phi_0, \tag{1}$$

There exist stationary solutions:

- bright soliton ($g_0 < 0$)

$$\phi_0(x-q) = \sqrt{\frac{N}{2\xi}}\,\frac{e^{-i\theta}}{\cosh\left(\frac{x-q}{\xi}\right)},$$

- dark soliton ($g_0 > 0$)

$$\phi_0(x-q) = e^{-i\theta}\sqrt{\rho_0}\,\tanh\left(\frac{x-q}{\xi}\right).$$

by Marc Haelterman

In our considerations dark soliton is placed in a weak external potential $V(x)$. To calculate a small perturbation of solitonic wavefunction we start with time–independent GPE:

$$-\frac{1}{2}\partial_x^2\phi(x) + \frac{1}{\rho_0}|\phi(x)|^2\phi(x) + V(x)\phi(x) = \mu\phi(x), \tag{2}$$

$$\phi = \phi_0 + \delta\phi \qquad\qquad \mu = \mu_0 + \delta\mu = 1 + \delta\mu.$$

$$\downarrow$$

Time–independent, non–homogeneous Bogoliubov–de Gennes equations:

$$\mathcal{L}\begin{bmatrix} \delta\phi \\ \delta\phi^* \end{bmatrix} = V\begin{bmatrix} -\phi_0 \\ \phi_0^* \end{bmatrix} + \delta\mu\begin{bmatrix} \phi_0 \\ -\phi_0^* \end{bmatrix}, \tag{3}$$

where

$$\mathcal{L} = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{2}{\rho_0}|\phi_0|^2 - 1 & +\frac{1}{\rho_0}\phi_0^2 \\ -\frac{1}{\rho_0}\phi_0^{*2} & \frac{1}{2}\partial_x^2 - \frac{2}{\rho_0}|\phi_0|^2 + 1 \end{pmatrix}. \tag{4}$$

$$\phi_0(x-q) = e^{-i\theta}\sqrt{\rho_0\xi}\,\tanh(x-q)\,.$$

J. Dziarmaga, Phys. Rev. A **70**, 063616 (2004)

Now we have all vectors to build a complete basis and deformation of the soliton can be expanded in that basis

$$\begin{bmatrix}\delta\phi\\ \delta\phi^*\end{bmatrix} = \Delta\theta\begin{bmatrix}u_\theta\\ v_\theta\end{bmatrix} + P_\theta\begin{bmatrix}u_\theta^{ad}\\ v_\theta^{ad}\end{bmatrix} + \Delta q\begin{bmatrix}u_q\\ v_q\end{bmatrix} + P_q\begin{bmatrix}u_q^{ad}\\ v_q^{ad}\end{bmatrix} + \sum_k\left(b_k\begin{bmatrix}u_k\\ v_k\end{bmatrix} + b_k^*\begin{bmatrix}v_k^*\\ u_k^*\end{bmatrix}\right). \tag{5}$$

$$\frac{P_\theta}{M_\theta} = -2\langle\partial_{N_0}\phi_0|V\phi_0\rangle + \delta\mu - iR(\langle u_q|V\phi_0\rangle + \langle v_q|V\phi_0^*\rangle) = 0$$

$$\swarrow \qquad\qquad \searrow$$

$$\delta\mu = 2\langle\partial_{N_0}\phi_0|V\phi_0\rangle \qquad\qquad \int_0^L dx|\phi(x-q)|^2\partial_x V(x) = 0$$

$$\downarrow$$

$$\delta\mu = \frac{1}{L}\int_0^L dy\,\left(\tanh y + y\,\mathrm{sech}^2 y\right)\tanh y V(y+q)$$

Let us start again with the stationary GPE but assume the solution we are looking for is a real function

$$\left(-\frac{1}{2}\partial_x^2 + \frac{1}{\rho_0}\phi^2 - \mu + V(x)\right)\phi = 0. \qquad (6)$$

$$\mu = \mu_0 + \delta\mu = 1 + \delta\mu \qquad \phi = \phi_0 + \delta\phi$$

$$\phi_0^2(x-q) = \rho_0 \tanh^2(x-q) = \rho_0\left[1 - \cosh^{-2}(x-q)\right]$$

$$\downarrow$$

$$x \to x + q$$

$$\left(H_0 + 2\right)\delta\phi = \delta\mu\phi_0 - V\left(x+q\right)\phi_0, \qquad (7)$$

where $H_0 = -\frac{1}{2}\partial_x^2 - \frac{3}{\cosh^2(x)}$ – Pöschl-Teller Hamiltonian.

There are two bound states: J. Lekner, Am. J. Phys. **75**, 1151 (2007)

$E_0 = -2$ $\psi_0(x) \sim \mathrm{sech}^2(x) \sim \partial_x \phi_0$

$E_1 = -\frac{1}{2}$ $\psi_1(x) \sim \mathrm{sech}(x)\mathrm{tanh}(x)$

and scattering states

$E_k = \frac{k^2}{2}$ $\psi_k(x)$

We can therefore expand deformation $\delta\phi$ over orthonormal basis of eigenfunctions

$$\delta\phi = \alpha_0\, \psi_0 + \alpha_1\, \psi_1 + \int dk\, \alpha_k\, \psi_k(x), \tag{8}$$

$$(E_j + 2)\alpha_j = \int dx\, \psi_j^*(x)[\delta\mu\phi_0 - V(x+q)\phi_0]. \tag{9}$$

In order to solve

$$(H_0 + 2)\,\delta\phi = \delta\mu\phi_0 - V\,(x+q)\,\phi_0, \tag{10}$$

we have to invert the operator $H_0 + 2$ in the Hilbert space what is simple because all eigenfunctions of H_0 are known. That is

$$\delta\phi(x) = \int dy\; K(x,y)\,[\delta\mu\phi_0 - V(y+q)\phi_0], \tag{11}$$

where the symmetric kernel $K(x,y)$ reads

$$\begin{aligned} K(x,y) &= \frac{2}{3}\psi_1(x)\psi_1^*(y) + 2\int \frac{\psi_k(x)\psi_k^*(y)}{4+k^2} \\ &= -\frac{1}{16}\mathrm{sech}^2(x)\mathrm{sech}^2(y)\times \\ &\quad \left\{\mathrm{sh}^2 2x + \mathrm{sh}^2 2y + 4\mathrm{ch}2x + 4\mathrm{ch}2y \right.\\ &\quad -3 - (\mathrm{ch}2x + \mathrm{ch}2y + 3)\,|\mathrm{sh}2x - \mathrm{sh}2y| \\ &\quad \left. -4\mathrm{sh}|x-y|\,\mathrm{sh}x\,\mathrm{sh}y - 6|x-y|\right\}. \end{aligned} \tag{12}$$

In the present approach the chemical potential has to be determined by the normalization condition $\langle\phi|\phi\rangle = N_0 + \mathcal{O}(\delta\phi^2)$

$$\begin{aligned} \delta\mu &= \frac{\int dxdy\ \phi_0(x)K(x,y)V(y+q)\phi_0(y)}{\int dxdy\ \phi_0(x)K(x,y)\phi_0(y)} \\ &= \frac{1}{L}\int dy\left(\tanh y + y\,\mathrm{sech}^2 y\right)\tanh y V(q+y), \end{aligned} \tag{13}$$

Expansion in Bogoliubov modes:

$$\phi(x) = \phi_0(x) + \sum_k \left[b_k u_k(x) + b_k^* v_k^*(x) \right] \tag{14}$$

Expansion in modes of the Pöschl–Teller potential:

$$\phi(x) = \phi_0(x) - \int dy\, K(x,y) V(y+q) \phi_0(y) + \delta\mu \, \frac{\partial \phi_0(x)|_{\mu_0}}{\partial \mu_0}, \tag{15}$$

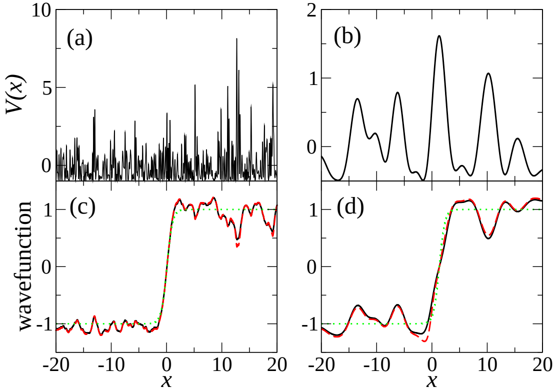

Figure: In panel (a) we show an example of the optical speckle potential with the correlation length $\sigma_R = 0.05$ and for $V_0 = 1$ while in panel (c) we present the corresponding solution of the Gross-Pitaevskii equation obtained numerically (solid black line) and within the perturbation approach (red dashed line). In panels (b) and (d) we show the same as in (a) and (c) but for $\sigma_R = 1$ and $V_0 = 0.5$. Green dotted lines in (c) and (d) correspond to unperturbed soliton wavefunctions.

Perturbative approach:

$$\begin{bmatrix} \phi \\ \phi^* \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_0^* \end{bmatrix} + \Delta\theta \begin{bmatrix} u_\theta \\ v_\theta \end{bmatrix} + P_\theta \begin{bmatrix} u_\theta^{ad} \\ v_\theta^{ad} \end{bmatrix} + \Delta q \begin{bmatrix} u_q \\ v_q \end{bmatrix} + P_q \begin{bmatrix} u_q^{ad} \\ v_q^{ad} \end{bmatrix} + \sum_k \left(b_k \begin{bmatrix} u_k \\ v_k \end{bmatrix} + b_k^* \begin{bmatrix} v_k^* \\ u_k^* \end{bmatrix} \right). \quad (16)$$

Non-perturbative Dziarmaga approach: J. Dziarmaga, Phys. Rev. A **70**, 063616 (2004)

$$\begin{bmatrix} \phi \\ \phi^* \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \phi_0^* \end{bmatrix} + P_\theta \begin{bmatrix} u_\theta^{ad} \\ v_\theta^{ad} \end{bmatrix} + P_q \begin{bmatrix} u_q^{ad} \\ v_q^{ad} \end{bmatrix} + \sum_k \left(b_k \begin{bmatrix} u_k \\ v_k \end{bmatrix} + b_k^* \begin{bmatrix} v_k^* \\ u_k^* \end{bmatrix} \right). \quad (17)$$

$\downarrow$ substitute

$$H = \int dx \left[\frac{1}{2} |\partial_x \phi|^2 + V|\phi|^2 + \frac{1}{2\rho_0} |\phi|^4 - \mu |\phi|^2 \right] \quad (18)$$

+ apply the second quantization formalism

The quantum effective Hamiltonian:

$$\hat{H} = \hat{H}_q + \hat{H}_B + \hat{H}_1, \tag{19}$$

where

$$\begin{aligned}
\hat{H}_q &= -\frac{\hat{P}_q^2}{2|M_q|} + \int dx V(x)|\phi_0(x-q)|^2 \\
&= -\left(\frac{\hat{P}_q^2}{2|M_q|} + \frac{|M_q|}{4}\int dx \frac{V(x)}{\cosh^2(x-q)}\right), \qquad (20) \\
\hat{H}_B &= \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k, \qquad (21) \\
\hat{H}_1 &= \sum_k (\langle u_k|V\phi_0\rangle + \langle v_k|V\phi_0^*\rangle)(\hat{b}_k + \hat{b}_k^\dagger). \qquad (22)
\end{aligned}$$

$$|\psi_n(q)|^2 \propto \exp\left(-\frac{|q-q_0|}{l_{loc}}\right) \qquad \text{Anderson -- localized eigenstates}$$

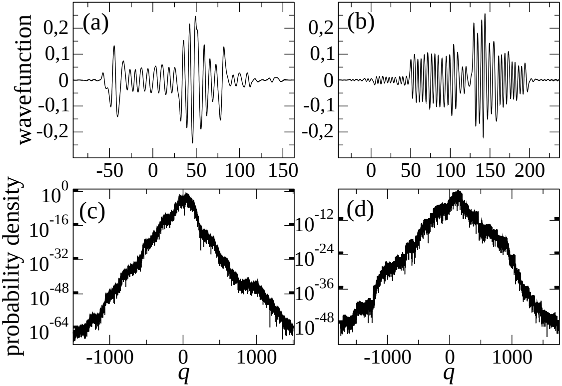

Figure: In top panels we show examples of eigenstates $|\psi_n(q)|^2$ of the effective Hamiltonian $\hat{H}_q$, see (20), while in bottom panels the corresponding probability densities in log scale. The correlation length of the speckle potential $\sigma_R = 0.28$ and the strength $V_0 = 7 \times 10^{-5}$ (left panels) and $V_0 = 1.4 \times 10^{-4}$ (right panels). The eigenstates correspond to the eigenvalue $E_n = -3.03 \times 10^{-3}$ (left panels) and $E_n = -8.58 \times 10^{-3}$ (right panels) and reveal the localization length $l_{loc} = 10.5$ and $l_{loc} = 15.7$, respectively.

$$|\Psi\rangle = |\psi_n, 0_B\rangle = \psi_n(q)|0_B\rangle, \quad (23)$$

According to the Fermi golden rule the decay rate reads

$$\Gamma = 2\pi \sum_m \gamma_m, \quad (24)$$

where

$$\gamma_m = \left|\langle\psi_m, 1_k|\hat{H}_1|\psi_n, 0_B\rangle\right|^2 g(\epsilon_k) = \left|\langle\psi_m|\langle u_k|V\phi_0\rangle + \langle v_k|V\phi_0^*\rangle\,|\psi_n\rangle\right| g(\epsilon_k)$$

The lifetime of the Anderson – localized states:

- $\tau = \frac{1}{\Gamma} = 8 \times 10^5$ (17 minutes) presented in Figs. 2a and 2c
- $\tau = 2.5 \times 10^5$ (5 minutes) presented in Figs. 2b and 2d

- Classical analysis of the dark soliton deformation show that external potential weakly affect soliton structure.
- Results obtained within perturbation approach and by numerical solution of the GPE reveal very good agreement even though the strength of the disorder is of the order of the chemical potential.
- We showed that the lifetime of the Anderson localized soliton is much longer than condensate lifetime in a typical experiment.

M. Mochol, M. Płodzień, K. Sacha, Phys. Rev. A **85**, 023627 (2012)