Limit to Spin Squeezing in BEC : from two-mode to multimode

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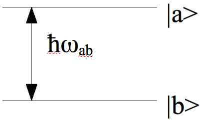
Plan

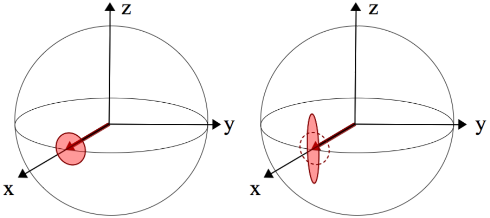
Spin squeezing and atomic clocks

N two-level atoms :



Collective spin :

$$S_x = \sum_j \left(|a\rangle\langle b| + |b\rangle\langle a|\right)_j /2,$$
$$S_z = \sum_j \left(|a\rangle\langle a| - |b\rangle\langle b|\right)_j /2$$



uncorrelated atoms squeezed

Uncorrelated atoms

$$\Delta\omega_{ab}^{\mathrm{unc}} = \frac{1}{\sqrt{NT}}$$

Squeezed state

$$\Delta\omega_{ab}^{\mathrm{sq}} = \xi \Delta\omega_{ab}^{\mathrm{unc}} = \frac{\xi}{\sqrt{NT}}$$

$$\xi^2 = \frac{N\Delta S_\perp^2}{\langle S_x\rangle^2}$$

Spin squeezing parameter

Kitagawa, Ueda, (1993) ; Wineland (1994)

Spin squeezing schemes in atomic ensembles

- **Light-Atoms interaction**

  Quantum Non Demolition measurement of S_z
  $\xi^2 = -3.0dB = 0.5$ **Vuletić PRL (2010)**

  $\xi^2 = -3.4dB = 0.46$ **Polzik J. Mod. Opt (2009)**

  Cavity feedback $\xi^2 = -10dB = 0.1$ **Vuletić PRL (2010)**

- **Interactions in BEC**

  Stationary method for BEC in two external states

  In a double well $\xi^2 = -3.8dB = 0.42$ **Oberthaler, Nature (2008)**

  In a double well on a chip **Reichel PRL (2010)**

  Dynamical method for BEC

  Feshbach $\xi^2 = -8.2dB = 0.15$ **Oberthaler, Nature (2010)**

  State-dependent pot. $\xi^2 = -2.5dB = 0.56$ **Treutlein, Nature (2010)**

Dynamical generation of spin squeezing in a BEC

- At $t < 0$ all the atoms are in condensate a. At $t = 0$, $\pi/2$-pulse
- Factorized state just after the pulse

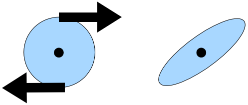
$$|x\rangle = \frac{1}{\sqrt{N!}} \left(\frac{a^\dagger + b^\dagger}{\sqrt{2}} \right)^N |0\rangle = \sum C_{N_a,N_b} |N_a, N_b\rangle$$

- Expansion of the Hamiltonian Castin, Dalibard PRA (1997)

$$\begin{aligned} \hat{H}(\hat{N}_a, \hat{N}_b) &= E(\bar{N}_\epsilon) + \mu_a(\hat{N}_a - \bar{N}_a) + \mu_b(\hat{N}_b - \bar{N}_b) \\ &+ \frac{1}{2}\partial_{N_a}\mu_a(\hat{N}_a - \bar{N}_a)^2 + \ldots \end{aligned}$$

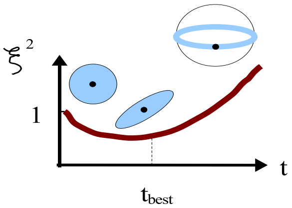
Non linear Hamiltonian

$$H_{NL} = \hbar\chi S_z^2$$



Dynamical generation of spin squeezing in a BEC

Best squeezing time



Predictions at $T = 0$ without decoherence :

$$H_{NL} = \hbar\chi S_z^2$$

$$\xi^2_{\text{best}} \sim \frac{1}{N^{2/3}} \qquad \chi t_{\text{best}} \sim \frac{1}{N^{2/3}}$$

No limit to the squeezing ?

Kitagawa, Ueda, PRA (1993) ; Sørensen et al. Nature (2001)

What limits spin squeezing for $N \to \infty$?

- **Particle losses** : **Li Yun, Y. Castin, A. Sinatra, PRL (2008)**

$$\min_{t,\omega,N} \xi^2 = \left[\left(\frac{5\sqrt{3}}{28\pi}\frac{m}{\hbar a}\right)^2 \left(\frac{7}{2}K_1 K_3\right)\right]^{1/3}$$

- **Non-zero temperature** : **A. Sinatra et al. PRL (2011) ; Frontiers of Phys. (Springer) (2011) ; Eur. Phys. Journ. D (2012)**

Spin squeezing scaling for $N \to \infty$

Uncorrelated atoms

$$\Delta\omega_{ab}^{\mathrm{unc}} \propto \frac{1}{\sqrt{N}}$$

Squeezed state

$$\Delta\omega_{ab}^{\mathrm{sq}} \propto \frac{\xi(N)}{\sqrt{N}}$$

Heisenberg limit

$$\Delta\omega_{ab}^{\mathrm{H}} \propto \frac{1}{N}$$

- **Two mode model** $H_{NL} = \hbar\chi S_z^2$ **Kitagawa Ueda**

$$N \to \infty, \quad \xi \sim \frac{1}{N^{1/3}} \quad \Rightarrow \quad \Delta\omega_{ab}^{\mathrm{sq}} \sim \frac{1}{N^{5/6}}$$

- Two mode model with **dephasing**
- Two mode model with **decoherence**
- Multimode description at **finite temperature** or zero temperature

$$N \to \infty, \quad \xi \sim \xi_{min} \neq 0 \quad \Rightarrow \quad \Delta\omega_{ab}^{\mathrm{sq}} \sim \frac{\xi_{min}}{\sqrt{N}}$$

Explicit calculations to obtain $\xi_{min}(\mathrm{dephasing})$, $\xi_{min}(\mathrm{losses})$, $\xi_{min}(\mathrm{temperature})$, ...

Two-mode dephasing model

HAMILTONIAN WITH A DEPHASING TERM

$$H = \hbar\omega_{ab}S_z + \hbar\chi\left(S_z^2 + DS_z\right)$$

G. Ferrini et al. PRA 2011, Sinatra et al. Frontiers of Physics 2012

D is a time-independent Gaussian random variable, $\langle D\rangle = 0$

$$\frac{\langle D^2\rangle}{N} \to \epsilon_{\mathrm{noise}}\ ; \qquad N \to \infty$$

Although the analytical solution holds $\forall\epsilon_{\mathrm{noise}}$, typically $\epsilon_{\mathrm{noise}} \ll 1$

- $\epsilon_{\mathrm{noise}} \Leftrightarrow$ **Fraction of lost particles**
- $\epsilon_{\mathrm{noise}} \Leftrightarrow$ **Non-condensed fraction** in the thermodynamic limit.

Spin dynamics and relative phase dynamics

$$a = e^{i\theta_a}\sqrt{N_a} \qquad [N_a, \theta_a] = i$$

$$b = e^{i\theta_b}\sqrt{N_b} \qquad [N_b, \theta_b] = i$$

$$a^\dagger b = \sqrt{N_a(N_b+1)}e^{-i(\theta_a-\theta_b)}$$

Initially : $N_a - N_b \sim \sqrt{N}$

and $\theta_a - \theta_b \sim \frac{1}{\sqrt{N}} \ll 1$

Spin components

$$S_x \simeq \frac{N}{2}\,; \qquad S_y \simeq -\frac{N}{2}(\theta_a - \theta_b)\,; \qquad S_z = \frac{N_a - N_b}{2}\,;$$

Heisenberg equation of motion for the phase difference

$$(\theta_a - \theta_b)(t) = (\theta_a - \theta_b)(0^+) - \chi t\,(2S_z + D)$$

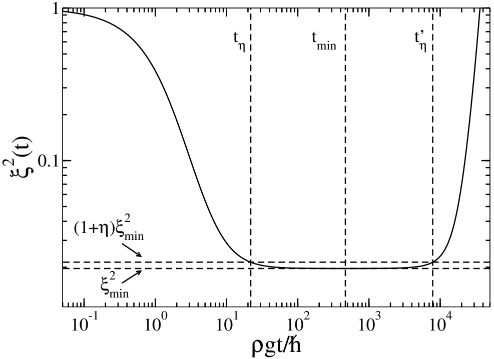
- **S_y becomes a copy of S_z** : squeezing as $\chi t \gg \frac{1}{N} \leftrightarrow \frac{\rho g t}{\hbar} \gg 1$
- Phase spreading $(\theta_a - \theta_b) \sim 1$ as $\chi t \simeq \frac{1}{\sqrt{N}} \leftrightarrow \frac{\rho g t}{\hbar} \gg \sqrt{N}$

Best spin squeezing and spin-squeezing time

$\xi^2_{\rm min}$ = minimum of ξ^2 over time

Best squeezing

$$\xi^2_{\rm min} \overset{N\to\infty}{\to} \frac{\langle D^2\rangle}{N} = \epsilon_{\rm noise}$$



Close to best squeezing time

$$\xi^2(t_\eta) = (1+\eta)\xi^2_{\rm min}$$

$$\frac{\rho g t_\eta}{\hbar} = \frac{1}{\sqrt{\eta \xi^2_{\rm min}}}$$

$$\frac{\rho g t_{\rm min}}{\hbar} \sim N^{1/4}$$

$$\frac{\rho g t'_\eta}{\hbar} \sim N^{1/2}$$

A different conclusion in the weak-dephasing limit

$$H = \hbar\chi\left(S_z^2 + \mathbf{D} S_z\right)$$

$$\langle D^2\rangle \to \textbf{constant}\ ; \qquad N \to \infty$$

(e.g. $N \to \infty$ at fixed non-condensed particles or lost particles)
cf. A. Sørensen PRA 2001

Best squeezing $$\xi^2_{\mathrm{min}} = \frac{3^{2/3}}{2}\frac{1}{N^{2/3}} + \frac{\frac{3}{2} + \langle D^2\rangle}{N} + o\left(\frac{1}{N}\right)$$

Best time $$\frac{\rho g t_{\mathrm{min}}}{\hbar} = 3^{1/6} N^{1/3} - \frac{\sqrt{3}}{4} + o(1)$$

We recover in this case the scaling of $H = \hbar\chi S_z^2$ plus corrections.

Particle losses: Monte-Carlo wave functions

- **Interaction picture with respect to $H_{\rm nl} = \hbar\chi S_z^2$**

$$c_a = e^{i\frac{H_{\rm nl}t}{\hbar}}\, a\, e^{-i\frac{H_{\rm nl}t}{\hbar}} \qquad\qquad c_b = e^{i\frac{H_{\rm nl}t}{\hbar}}\, b\, e^{-i\frac{H_{\rm nl}t}{\hbar}}$$

- **Effective Hamiltonian and Jump operators for m-body losses**

$$H_{\rm eff} = -\sum_{\epsilon=a,b} \frac{i\hbar}{2}\gamma^{(m)} c_\epsilon^{\dagger m} c_\epsilon^m \qquad\qquad S_\epsilon = \sqrt{\gamma^{(m)}}\, c_\epsilon^m$$

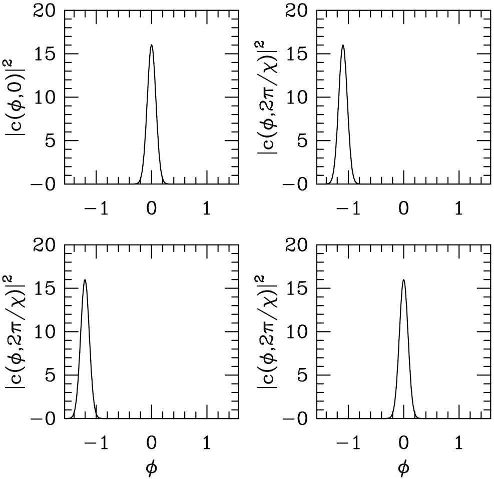
- **Evolution of one wave function with k jumps**

$$|\psi(t)\rangle = e^{-iH_{\rm eff}(t-t_k)/\hbar} S_{\epsilon_k} e^{-iH_{\rm eff}\tau_k/\hbar} S_{\epsilon_{k-1}} \dots S_{\epsilon_1} e^{-iH_{\rm eff}\tau_1/\hbar}|\psi(0)\rangle$$

- **Quantum averages**

$$\langle\hat{\mathcal{O}}\rangle = \sum_k \int_{0<t_1<t_2<\cdots t_k<t} dt_1 dt_2 \cdots dt_k \sum_{\{\epsilon_j\}} \langle\psi(t)|\hat{\mathcal{O}}|\psi(t)\rangle$$

Jumps randomly kick the relative phase



Relative phase distribution at $t = 0$ and $\chi t = 2\pi$ in single Monte Carlo realizations with 3, 1 and 0 quantum jumps
Sinatra, Castin EPJD 1998

$$c_a(t)|\phi\rangle_N \propto |\phi - \chi t/2\rangle_{N-1}$$

$$c_b(t)|\phi\rangle_N \propto |\phi + \chi t/2\rangle_{N-1}$$

After k jumps $|\psi(t)\rangle \propto |\phi + \frac{\chi t}{2}\mathcal{D}\rangle_{N-k}$ **with** $\mathcal{D} = \frac{1}{t}\sum_{l=1}^{k} t_l \left(\delta_{\epsilon_l,b} - \delta_{\epsilon_l,a}\right)$

N.B. : $e^{-\frac{i}{\hbar}\chi D S_z t}|\phi\rangle = |\phi - \frac{\chi t}{2} D\rangle$

Best squeezing and best time for $N \to \infty$

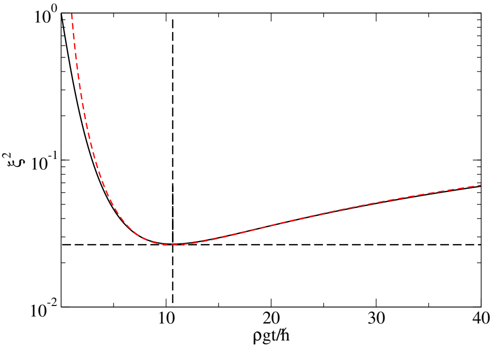
We use the exact solution for one-body losses :
γt = **fraction of lost particles at time** t

$$N \to \infty \qquad \gamma t \equiv \epsilon_{\mathrm{loss}} = \mathrm{const} \ll 1$$

For long times $\frac{\rho g t}{\hbar} \gg 1$

$$\xi^2(t) \simeq \frac{\langle \mathcal{D}^2 \rangle}{N} + \left(\frac{\hbar}{\rho g t}\right)^2 [1 + O(\gamma t)]$$

$$\frac{\langle \mathcal{D}^2 \rangle}{N} \simeq \frac{\gamma t}{3}$$



$$\xi^2_{\min} = \frac{3}{4}\left(\frac{4}{3}\frac{\hbar\gamma}{\rho g}\right)^{2/3}$$

$$\frac{\rho g t_{\min}}{\hbar} = \frac{1}{\sqrt{\frac{4}{3}\xi^2_{\min}}}$$

Unified view between *dephasing* noise and *losses*

Particle Losses	Dephasing model
$\lvert\psi(t)\rangle \propto \lvert\phi + \frac{\chi t}{2}\mathcal{D}\rangle$	$(\theta_a - \theta_b)(t) = (\theta_a - \theta_b)(0^+) - \chi t\,[2S_z + D]$
$\mathcal{D}$ from quantum jumps	D from a dephasing H
$\xi^2(t) \underset{\rho g t/\hbar > 1}{\simeq} \frac{\langle \mathcal{D}^2 \rangle}{N}$	$\xi^2(t) \underset{\rho g t/\hbar > 1}{\simeq} \frac{\langle D^2 \rangle}{N}$
$\frac{\langle \mathcal{D}^2 \rangle}{N} = \frac{\gamma t}{3} = \frac{\epsilon_{\mathrm{loss}}}{3}$	$\frac{\langle D^2 \rangle}{N} = \epsilon_{\mathrm{noise}}$

Multimode description

Hamiltonian for component a (idem for b)

$$H = dV \sum_{\mathbf{r}} \psi_a^\dagger(\mathbf{r}) h_0 \psi_a(\mathbf{r}) + \frac{g}{2} \psi_a^\dagger(\mathbf{r}) \psi_a^\dagger(\mathbf{r}) \psi_a(\mathbf{r}) \psi_a(\mathbf{r}) \,.$$

Before the pulse, the system is in thermal equilibrium in a with $T \ll T_c$.

the pulse mixes the field a with the field b that is in vacuum :

$$\psi_a(\mathbf{r})(0^+) = \frac{\psi_a(\mathbf{r})(0^-) - \psi_b(\mathbf{r})(0^-)}{\sqrt{2}}$$

After the pulse the two fields evolve independently

Bogoliubov description

Bogoliubov expansion : weakly interacting quasi-particles

$$H_a = E_0 + \sum_{\mathbf{k}\neq 0} \epsilon_k c^\dagger_{a\mathbf{k}} c_{a\mathbf{k}} + \textbf{cubic terms} + \textbf{quartic terms}$$

Spin components

$$S_+ \equiv S_x + iS_y = dV \sum_{\mathbf{r}} \psi_a^\dagger(\mathbf{r})\psi_b(\mathbf{r}) \qquad S_z = \frac{N_a - N_b}{2}$$

In the Bogoliubov description

$$S_+ = e^{i(\theta_a - \theta_b)} \left(\frac{N}{2} + F \right)$$

$$(\theta_a - \theta_b)(t) = (\theta_a - \theta_b)(0^+) - \frac{gt}{\hbar V} \left[(N_a - N_b) + \mathbf{D} \right]$$

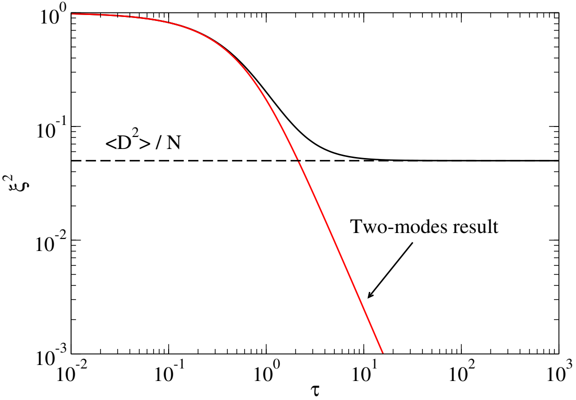
D and F depend on Bogoliubov functions and occupation numbers of quasi particles $c^\dagger_{a\mathbf{k}} c_{a\mathbf{k}}$ after the pulse

Squeezing parameter evolution

Double expansion in $\epsilon_{\rm size} = 1/N \to 0$ and $\epsilon_{\rm Bog} = \langle N_{\rm nc} \rangle / N \to 0$.

Spin squeezing saturates to a finite value

Spin squeezing as a function of a renormalized time ($\tau \simeq \rho g t / (2\hbar)$)

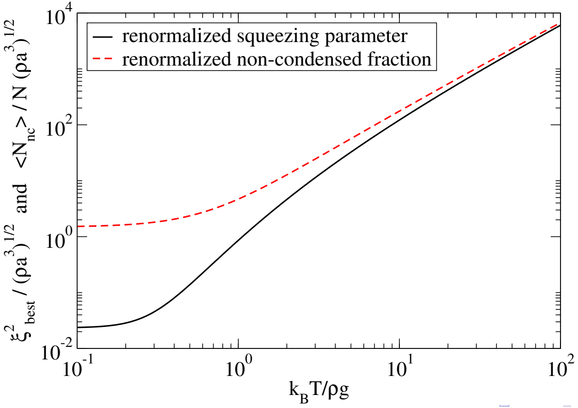


The limit $\langle D^2 \rangle / N$ depends on temperature and interaction strength

The limit of spin spin squeezing is smaller than the non condensed fraction

$$\xi^2_{\rm best} = \frac{\langle \mathbf{D}^2 \rangle}{N} = \sqrt{\rho a^3} \quad F\left(\frac{k_B T}{\rho g}\right)$$

Spin squeezing and the non condensed fraction both divided by $\sqrt{\rho a^3}$

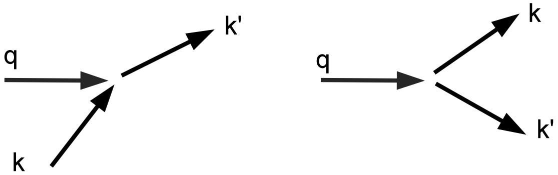


Unified view between *dephasing* noise and *temperature*

Dephasing model	Multimode $T \neq 0$
$(\theta_a - \theta_b)(t) \simeq -\chi t\,[2S_z + D]$	$(\theta_a - \theta_b)(t) \simeq -\chi t\,[2S_z + D_{\mathrm{th}}]$
D from a dephasing H	D_{th} from excited modes population
$\xi^2(t) \underset{\rho g t/\hbar > 1}{\simeq} \frac{\langle D^2 \rangle}{N}$	$\xi^2(t) \underset{\rho g t/\hbar > 1}{\simeq} \frac{\langle D_{\mathrm{th}}^2 \rangle}{N}$
$\frac{\langle D^2 \rangle}{N} = \epsilon_{\mathrm{noise}}$	$\frac{\langle D_{\mathrm{th}}^2 \rangle}{N} = \sqrt{\rho a^3}\, F(k_B T/\rho g) \underset{k_B T > \rho g}{\sim} \epsilon_{\mathrm{Bog}}$

Consequence of the physics beyond Bogoliubov approximation

$$H_a = E_0 + \sum_{\mathbf{k}\neq\mathbf{0}} \epsilon_k c^{\dagger}_{a\mathbf{k}} c_{a\mathbf{k}} + \textbf{cubic terms} + \textbf{quartic terms}$$

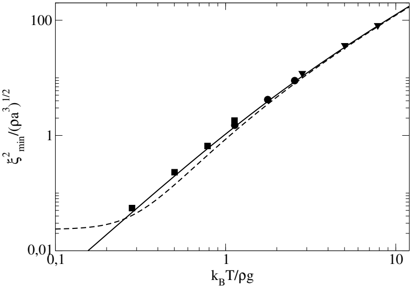


At long time the system thermalizes and Bogoliubov approximation fails

To test the validity of the perturbative treatment, we compare the analytic results with classical field simulations

Analytics versus Numerics (non perturbative)

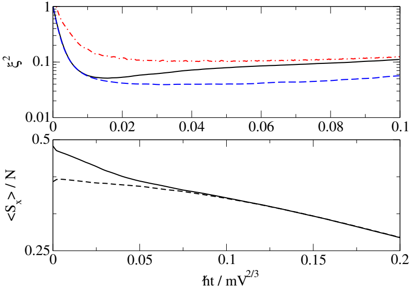
Best squeezing



$$\xi^2_{\mathrm{best}} = \frac{\langle \mathbf{D}^2 \rangle}{N} = \sqrt{\rho a^3} \quad F\left(\frac{k_B T}{\rho g}\right)$$

PRL (2011), long : EPJ ST (2012)

Thermalization in simulations



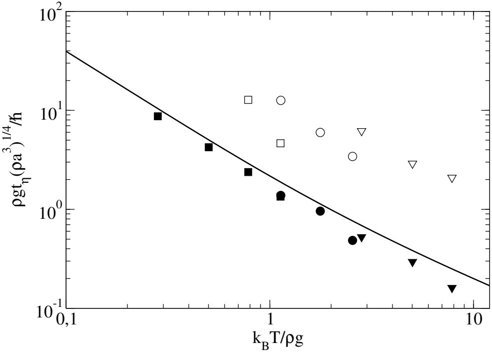
$$\langle S_x \rangle = \mathrm{Re}\,\langle \textstyle\sum_{\mathbf{k}} b^*_{\mathbf{k}} a_{\mathbf{k}} \rangle \underset{t>t_{\mathrm{therm}}}{\simeq} \mathrm{Re}\,\langle b^*_{\mathbf{0}} a_{\mathbf{0}} \rangle \,.$$

Result : Close to best squeezing time

At the thermodynamic limit, in the perturbative approach, $t_{\rm best} = \infty$.

Definition : $\xi^2(\mathbf{t}_\eta) = (\mathbf{1}+\eta)\xi^2_{\rm best}$

$$\frac{\rho g}{\hbar} t_\eta = \frac{1}{\sqrt{\eta \xi^2_{\rm best}}}$$



Necessary condition

$$\mathbf{t}_\eta \ll \mathbf{t}_{\rm therm}$$

One can show that

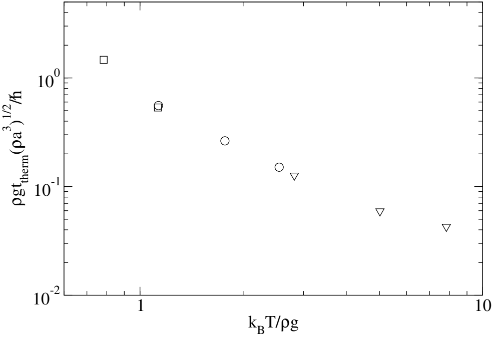
$$\frac{\mathbf{t}_\eta}{\mathbf{t}_{\rm therm}} \propto (\rho \mathbf{a}^3)^{1/4}$$

Rescaled thermalization time

At the thermodynamic limit, in the perturbative approach, $t_{\text{best}} = \infty$.

Definition : $\xi^2(\mathbf{t}_\eta) = (\mathbf{1} + \eta)\xi^2_{\text{best}}$

$$\frac{\rho g}{\hbar} t_\eta = \frac{1}{\sqrt{\eta \xi^2_{\text{best}}}}$$



NECESSARY CONDITION

$$\mathbf{t}_\eta \ll \mathbf{t}_{\text{therm}}$$

ONE CAN SHOW THAT

$$\frac{\mathbf{t}_\eta}{\mathbf{t}_{\text{therm}}} \propto (\rho \mathbf{a}^3)^{1/4}$$

Physical Interpretation

$$(\theta_a - \theta_b) = -\frac{g}{\hbar V}\, t\, [N_a - N_b + \mathcal{D}]$$

Limit to spin squeezing

$$\mathbf{D} \neq \mathbf{0} \quad \Rightarrow \quad \xi^2 = \frac{\langle \mathbf{D}^2 \rangle}{N} \neq 0 \quad \text{pour} \quad N \to \infty$$

From where this dephasing comes from ?

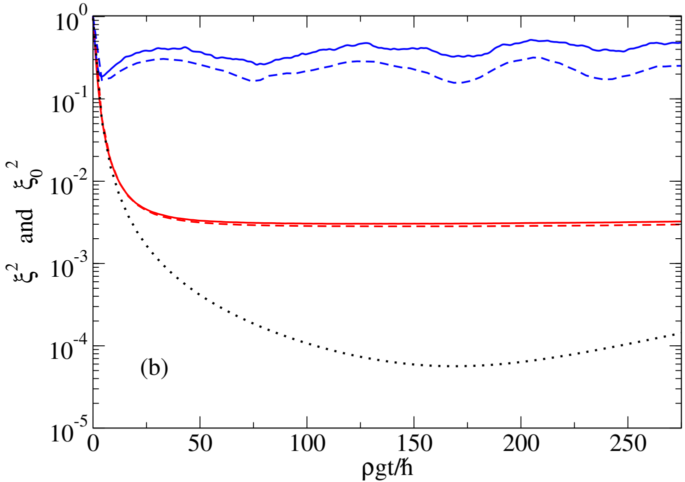
Hartree-Fock limit $k_B T \gg \rho g$, $\mathbf{D} = \mathbf{N_{a\perp}} - \mathbf{N_{b\perp}}$ (and $\langle D^2 \rangle = N_{nc}$):

$$(\theta_a - \theta_b)_{HF} = -\frac{g}{\hbar V}\, t\, [N_{a\mathbf{0}} - N_{b\mathbf{0}} + (1 + \mathbf{1})(N_{a\perp} - N_{b\perp})]$$

condensate + condensate $\leftrightarrow g$

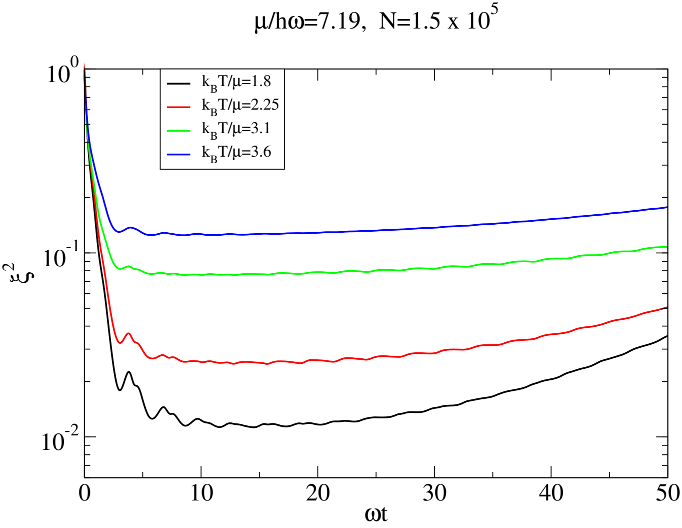
condensate + non condensate $\leftrightarrow 2g$

Condensate squeezing vs Total field squeezing



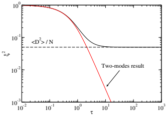
$k_B T/\rho g = 0.5$, $\langle N_{\rm nc}\rangle/N = 0.02$, $\sqrt{\rho a^3} = 1.32 \times 10^{-2}$.

Numerical results in the trap : squeezing as a function of time



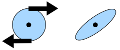
Conclusions

- **Spin squeezing** *with dephasing*, *with losses*, or in a *multimode theory* at $T \neq 0$ is **limited for** $N \to \infty$. We calculate this limit microscopically.



- A simple **dephasing model** can effectively describe both the *lossy* and *finite temperature* case. In both cases the limit is given by a **fluctuating perturbation of the relative phase**.

$$S_y \propto 2S_z + \mathbf{D}$$



- In the case at finite temperature the perturbation comes from **thermal population** of the excited modes and from the **different interaction strength** for c-c atoms and c-nc atoms.
- Condensate squeezing is much worse than the squeezing of the total field.